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A Mixed Spectral/Wavelet Method for the Solution of the Stokes Problem

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The paper presents a mixed wavelet/spectral Chebychev method for solving the unsteady 2D Stokes equations in the vorticity-stream function formulation with periodicity condition in one direction. After an appropriate time discretisation of the equations, one has to solve at each time step a stationary Stokes-like problem. A capacitance matrix method is used to eliminate the problem of boundary conditions. This leads to solving a series of Helmholtz problems. The spatial discretisation makes use of the wavelet method in the periodic direction and the spectral collocation Chebychev method in the non-periodic direction. The resolution of the discrete Helmholtz problem is done by means of the diagonalisation technique in the non-periodic direction. The system then splits into a sequence of one dimensionnal periodic Helmholtz problems which are efficiently inverted using FFTs. Numerical tests show both the stability and the accuracy of the method. © 1998 Academic Press

1. INTRODUCTION

In this paper we develop a method which combines the wavelet method in one direction and the spectral Chebychev method in another direction for solving incompressible unsteady Stokes equations. There have been numerous computations of incompressible flow using mixed methods such as finite difference/spectral methods which can be found in the literature. In a framework strictly spectral, when periodicity conditions are assumed in some coordinates space, a mixed spectral Fourier/Chebychev method that uses Fourier expansion in the periodic direction and the Chebychev method in the other direction is commonly used. Recently, a new numerical concept was introduced and is gaining increasing popularity. The method is based on the expansion of functions in terms of a set of basis functions called wavelets. Wavelets are a new family of functions which constitute a basis of $\mathbf{L}^2(\mathbf{R})$. They have many attractive features: orthogonality, compact support, arbitrary

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regularity, and simplicity since they are obtained by dilation and translation of a single function. In data analysis where it was first applied, the wavelet transform was found to give better results than the classical Fourier transform. Wavelets combine the advantage of both finite-difference (or finite elements) and spectral methods: good localisation and spectral accuracy in regard to the degree of regularity of the basis functions. Therefore one expects that the wavelets method will be well suited for situations where classical methods such as finite differences do not converge and where Fourier method does not apply.

The flow is assumed to be bidimensional, with periodicity conditions in one direction. The equations are considered in the vorticity-stream function formulation. This formulation automatically satisfies the incompressibility condition and allows one to reduce the number of equations to be solved. However, the nature of the boundary conditions is troublesome since they imply the specification of both the stream function and its normal derivative but none for the vorticity. Various ways are usually used to tackle this difficulty; the common way consists of deriving boundary conditions for the vorticity by manipulating the Neumann boundary condition $\partial_{\mathbf{n}}\psi|_{\Gamma}$ and the relation $\omega = -\nabla^2\psi$. This technique has been used in finite differences methods or finite elements methods as well as in spectral methods (cf. Quartapelle [6] and Weinan *et al.* [5] for discussions on the basic issues on the numerical treatment of the vorticity-stream function equations). Glowinski and Pironneau [17] studied the relation between the trace of ω and the normal derivative of ψ on the boundary and they introduced a treatment of boundary conditions for ω. Using a finite elements approximation they deduced a linear system $A\omega|_{\Gamma} = \partial_{\mathbf{n}}\psi|_{\Gamma}$ for the trace of ω on the boundary via the solutions of Dirichlet problems for $-\nabla^2$. The method was further perfected by Dean *et al.* [18]. This treatment of boundary conditions for ω is usually called the influence matrix method. In spectral approximation the influence matrix (Ehrenstein and Peyret [12]) is the method most frequently used for solving the equations of vorticity-stream function. An alternative to the influence matrix method is the vorticity integral method in which boundary conditions are derived for the vorticity by using Green's identities (Quartapelle, [6]; Nguyen *et al.* [8]). In the present wavelet/spectral method, we use the influence matrix method to solve the problem. The influence matrix, also called the capacitance matrix method, has been widely used to solve linear elliptic problems where boundary conditions present some difficulties. For instance it is commonly associated with the domain imbedding technique for solving problems in complex geometries (Garba [2]). It is also used for situations where boundary conditions are not available for all the unknowns, particularly in incompressible flow calculations. In this context, the influence matrix method has been used first by Kleiser and Schumann [7] in the spectral calculation of 3D flow in primitive variables with two directions of periodicity. Later the method was extended by Le Quéré and Aziary de Rocquefort [10] and Tuckerman [9] for situations with more complicated boundary conditions. In the vorticity-stream function formulation, Vanel *et al.* [11] and Ehrenstein and Peyret [12] developed a capacitance matrix method to overcome the lack of boundary conditions for the vorticity. The capacitance matrix we use here is similar to that presented Ehrenstein and Peyret [12].

In Section 2, we present the Stokes-equations and their time discretisation. A three-level scheme which provides a second order accuracy is used to discrete the problem. Then at each time step, one has to solve a stationary Stokes-type problem.

The method for solving this problem is presented in Section 4. It makes use of the influence matrix technique. The influence matrix method leads to solving a series of Helmholtz problems with periodicity condition in one Direction and Dirichlet boundary conditions in the other direction.

Section 3 is devoted to the numerical resolution of the Helmholtz problem. In the direction of periodicity, the discretisation is done in the basis constituted by the translates and the dilations of the Daubechies scaling function. We present two methods for the wavelet discretisation: the first method is of collocation kind (Garba [3]) and the second is based on the Galerkin method (Amaratunga and Williams [4] and Qian and Weiss [19, 20]). In the non-periodic direction the discretisation is done in the collocation Chebychev method. The system of equations arising from the wavelet/collocation Chebychev discretisation is inverted by applying first the diagonalisation technique in the non-periodic direction. The problem is then split into a series of one-dimensional Helmholtz equations discretised in the wavelet method. The solution of these problems is efficiently obtained by resorting to FFTs.

2. THE STOKES EQUATIONS

The flow is assumed to be bidimensional in the plane (x, y) and periodic in the x direction. The unsteady Stokes equations can be conveniently written in the vorticity-stream function formulation as

$$\partial_t \omega - \nu \nabla^2 \omega = f \qquad \text{in } D \tag{1}$$

$$\nabla^2 \psi + \omega = 0 \qquad \text{in } D \tag{2}$$

with boundary conditions in the y direction given by

$$\psi = g \qquad \text{on } \Gamma \tag{3}$$

$$\partial_y \psi = h \qquad \text{on } \Gamma \tag{4}$$

and periodic boundary conditions in the x direction.

The constant ν in (1) is the viscosity. The vorticity ω and the stream function ψ are related to the velocity field $V = (u, v)$ by

$$\omega = \partial_x v - \partial_y u, \qquad u = \partial_y \psi, \qquad v = -\partial_x \psi. \tag{5}$$

From the initial condition, $V(0) = V_0$ prescribed for the velocity, an initial condition deduced for the vorticity variable $\omega(t = 0) = \omega_0 = \partial_x v_0 - \partial_y u_0$. The domain D and boundary Γ are defined by

$$D = \{(x, y), 0 \leq x \leq 1, -1 \leq y \leq 1\} \tag{6}$$

$$\Gamma = \{(x, -1), 0 \leq x \leq 1\} \cup \{(x, 1), 0 \leq x \leq 1\}. \tag{7}$$

The vorticity equation (1) is discretised in time through the scheme

$$\frac{3\omega^{n+1} - 4\omega^n + \omega^{n-1}}{\Delta t} - \nu \nabla^2 \omega^{n+1} = f^{n+1}, \tag{8}$$

which provides a second order accuracy in time. Equation (2) of the stream function and the boundary conditions (3)–(4) are discretised in an implicit manner

$$\nabla^2\psi^{n+1} + \omega^{n+1} = 0 \tag{9}$$

$$\psi^{n+1} = g^{n+1} \tag{10}$$

$$\partial_y\psi^{n+1} = h^{n+1}. \tag{11}$$

The parameter Δt in (8) denotes the time step and the quantity ϕ^n stands for the approximation of the function ϕ at time $t_n = n\Delta t$. Then at each time step one has to solve a stationary Stokes-like problem

$$\nabla^2\omega^{n+1} - \sigma\omega^{n+1} = f^{n,n-1} \qquad \text{in } D \tag{12}$$

$$\nabla^2\psi^{n+1} + \omega^{n+1} = 0 \qquad \text{in } D \tag{13}$$

with boundary conditions (10)–(11) and periodicity condition in x.

In (12) the parameter $\sigma = 3/(2\Delta t)$ and the right hand side $f^{n,n-1}$ contains the forcing term f^{n+1} and all the quantities coming from the previous time steps $(n-1)\Delta t$ and $n\Delta t$.

The main difficulty in solving the problem (10)–(13) numerically arises from the nature of the boundary conditions: two boundary conditions are prescribed for the stream function while no boundary conditions are available for the vorticity. We use the capacitance matrix method to circumvent this difficulty.

3. THE HELMHOLTZ SOLVER

Because the numerical solution of the Stokes-like problem (10)–(13) entails solving a sequence of Helmholtz problems, we present first the numerical method for solving the Helmholtz problem. Let us consider the bidimensionnal Helmholtz equation

$$\nabla^2 u - \sigma u = f, \qquad (x, y) \in D, \tag{14}$$

where σ is a non-negative constant and the domain D is defined by (6). We assume periodic boundary conditions in the x-direction and in the y-direction we consider the mixed Dirichlet–Neumann boundary conditions

$$\alpha_- u(x, -1) + \beta_-\partial_y u(x, -1) = g^-(x) \tag{15}$$

$$\alpha_+ u(x, 1) + \beta_+\partial_y u(x, 1) = g_+(x), \tag{16}$$

where the coefficients α_-, α_+, β_-, and β_+ satisfy the conditions

$$\alpha_-\beta_- \geq 0, \qquad \alpha_+\beta_+ \geq 0. \tag{17}$$

3.1. *Numerical Discretisation*

The numerical approximation of the Helmholtz problem (14)–(16) makes use of the wavelet method in the direction of periodicity and the collocation Chebychev method in the non-periodic direction. The family of Daubechies wavelets is used to implement the method.

The approximation is done on the basis of the translates and dilations of the Daubechies scaling function.

The Daubechies correlation function (Daubechies [1]) with N non-vanishing "filter coefficients" satisfies the scaling relation

$$\varphi(x) = \sum_{k=0}^{N-1} h_k \varphi(2x - k). \tag{18}$$

The parameter N will be referred to as the degree of the scaling function φ. In relation (18) the "filter coefficients" h_k, $k = 0, \ldots, N-1$, are chosen so that the scaling function φ has some desirable properties. The function φ has support in interval $[0, N-1]$ and it induces a multiresolution analysis on $\mathbf{L}^2(\mathbf{R})$, i.e., a nested sequence of functional spaces V_j, $j \in \mathbf{Z}$, such that the union is dense in $\mathbf{L}^2(\mathbf{R})$, and for each j, the sequence $\{\varphi_{j,k}\}_{k\in\mathbf{Z}}$ defined by

$$\varphi_{j,k}(x) = 2^{j/2}\varphi(2^{j/2}x - k) \tag{19}$$

forms an orthogonal Riesz basis for V_j. Now choosing an approximation space V_J, a function u belonging to $\mathbf{L}^2(\mathbf{R})$ is expanded in the basis (19) at the scale J,

$$u_J(x) = \sum_{k\in\mathbf{Z}} c_k \varphi_{J,k}(x). \tag{20}$$

The coefficients $\{c_k\}_{k\in\mathbf{Z}}$ define the function u_J in the wavelets space. We will use the appellation "wavelets coefficients" to designate these coefficients even though the approximation uses the scaling functions basis.

Now one needs some techniques for evaluating the wavelets coefficients. In the periodic situation this can be done by considering the values of the function at a set of discrete points $2^{-J}k$, $k = 0, \ldots, 2^J - 1$. Writing the expansion (20) at these points, and taking into account the periodicity condition, one gets a linearly independent system of order 2^J which links the wavelet coefficients to the values of the function u_J at the discrete points.The operator of the system is a circulant operator with kernel $K_\varphi = (0, \varphi_1, \ldots, \varphi_{N-2}, 0, \ldots, 0)$, where $\varphi_i = \varphi(i)$. Thus the physical values of the function are obtained by taking the convolution product of the kernel K_φ and the vector of the wavelet coefficients

$$K_\varphi * C = U \tag{21}$$

since multiplication of a vector by a circulant matrix is the same as a convolution between the vector and the first column of the matrix. The convolution product in (21) may be efficiently done by resorting to FFTs. For this we first take the Fourier transform of (21) to obtain

$$\mathcal{F}_k(K_\varphi) \cdot \mathcal{F}_k(C) = \mathcal{F}_k(U) \tag{22}$$

since a convolution in physical space is equivalent to a term by term product in Fourier space. The notation $\mathcal{F}_k$ is used for the coefficients in the Fourier space. Conversely one gets the wavelet coefficients from the physical values by

$$\mathcal{F}_k(C) = \mathcal{F}_k(U)/\mathcal{F}_k(K_\varphi). \tag{23}$$

Remark. All the Fourier coefficients $\mathcal{F}_k(K_\varphi)$ are different from zero by virtue of the reversibility of the discrete transformation (21).

Now coming back to the bidimensional problem (14), we use the wavelet basis described above to approximate the solution in the periodic direction and in the non-periodic direction we use the Chebychev approximation.

As in the previous setting, the scale used is J. The function $u(x, y)$ is expanded in the x direction in the basis (19)

$$u(x, y) = \sum_{k\in\mathbf{Z}} \phi_k(y)\varphi_{J,k}(x), \tag{24}$$

where this time the wavelet coefficients ϕ_k depend upon the variable y. It is convenient to make the variable transform $z = 2^J x$ so that the expansion (24) becomes

$$u(z, y) = 2^{J/2} \sum_{k\in\mathbf{Z}} \phi_k(y)\varphi(z-k). \tag{25}$$

In the same way, the forcing term f is expanded in the basis (19). Now substituting the expansion of u and f in Eq. (14) one gets

$$\begin{aligned} \sum_{k\in\mathbf{Z}} 2^{2J}\phi_k(y)\varphi''(z-k) + \sum_{k\in\mathbf{Z}} \phi_k''(y)\varphi(z-k) - \sigma \sum_{k\in\mathbf{Z}} \phi_k(y)\varphi(z-k) \\ = \sum_{k\in\mathbf{Z}} \hat{f}_k(y)\varphi(z-k). \end{aligned} \tag{26}$$

The same substitution is done in the boundary conditions (15) and (16) to obtain respectively

$$\alpha_- \sum_{k\in\mathbf{Z}} \phi_k(-1)\varphi(z-k) + \beta_- \sum_{k\in\mathbf{Z}} \phi'(-1)\varphi(z-k) = \sum_{k\in\mathbf{Z}} \hat{g}_k^- \varphi(z-k) \tag{27}$$

$$\alpha_+ \sum_{k\in\mathbf{Z}} \phi_k(1)\varphi(z-k) + \beta_+ \sum_{k\in\mathbf{Z}} \phi'(1)\varphi(z-k) = \sum_{k\in\mathbf{Z}} \hat{g}_k^+ \varphi(z-k). \tag{28}$$

Thus the projection of (14)–(16) in the wavelet space results in the system of Eqs. (26)–(28). We use two different techniques to discretise this set of equations in the z variable: the wavelet collocation method and the wavelet Galerkin method. Both methods are then coupled with a collocation Chebychev discretisation in the y-direction. The mixed wavelet collocation/collocation Chebychev will be referred to as the WC/CC method and the wavelet Galerkin/collocation Chebychev as the WG/CC method. These techniques are presented in the following subsections.

3.1.1. The wavelet collocation/Chebychev collocation method. We present here the discretisation of (26)–(28) in the WC/CC method. First the interval [0, 1] is discretised into the dyadic points

$$x_i = 2^{-J} i, \qquad i = 0, \ldots, N_x - 1, \tag{29}$$

where $N_x = 2^J$. The collocation method consists in writing the equations at the set of discrete points x_i, which amounts to writing Eqs. (26)–(28) at the integer points $z_i = i$, $i = 0, \ldots, N_x - 1$. Here we don't need to expand the forcing term in the wavelet space in Eqs. (26)–(28) since only their physical values are needed. Expressing the differential equations (26) at the integer points, one gets the semi-discrete system of equations

$$4^J \sum_{k\in\mathbf{Z}} \phi_k(y)\varphi''_{i-k} + \sum_{k\in\mathbf{Z}} \phi''_k(y)\varphi_{i-k} - \sigma \sum_{k\in\mathbf{Z}} \phi_k(y)\varphi_{i-k} = f(x_i, y), \tag{30}$$

where $\varphi_{i-k} = \varphi(i-k)$ and $\varphi''_{i-k} = \varphi''(i-k)$.

In a similar way the boundary conditions (27) and (28) are written at the integer points z_i, $0 \le i \le N_x - 1$, to obtain respectively

$$\alpha_- \sum_{k\in\mathbf{Z}} \phi_k(1)\varphi_{i-k} + \beta_- \sum_{k\in\mathbf{Z}} \phi'(1)\varphi_{i-k} = g^-(x_i, -1) \tag{31}$$

and

$$\alpha_+ \sum_{k\in\mathbf{Z}} \phi_k(1)\varphi_{i-k} + \beta_+ \sum_{k\in\mathbf{Z}} \phi'(1)\varphi_{i-k} = g^+(x_i, 1). \tag{32}$$

The system (30) together with the boundary conditions (31)–(32) constitutes a system of second order ordinary differential equations for the wavelets coefficients ϕ_i. We now introduce the discretisation of these equations in y. This is done in the Chebychev collocation method. The interval $[-1, 1]$ is discretised into the collocation points

$$y_j = \cos(j\pi/N_y), \qquad j = 0, \ldots, N_y. \tag{33}$$

The points (33) are the Gauss–Lobatto points. They are the collocation points most frequently used in spectral Chebychev methods because they not only guarantee a good convergence but also allow the use of FFTs.

We are then looking for a polynomial of degree N_y

$$\phi(y) = \sum_{j=0}^{N_y} \hat{\phi}_j T_j(y),$$

where T_j is the Chebychev polynomial of degree j. For a non-negative integer n the discrete values of derivatives $\phi^{(n)}$ are related to the discrete values of the function ϕ by the relation

$$\phi^{(n)}(y_j) = \sum_{l=0}^{N_y} d^{(n)}_{j,l}\phi(y_l). \tag{34}$$

The coefficients $d^{(n)}_{j,l}$ are the entries of the discrete approximation for the derivative of order n in the collocation Chebychev method. Then writing (30) at the collocation points y_j and substituting and the second derivative ϕ'' with its expression one gets

$$\sum_{k\in\mathbf{Z}} \{4^J \varphi''_{i-k} - \sigma\varphi_{i-k}\}\phi_{k,j} + \sum_{k\in\mathbf{Z}} \sum_{l=0}^{N_y} d^{(2)}_{j,l}\phi_{k,l}\varphi_{i-k} = f_{i,j},$$
$$0 \le i \le N_x - 1, 0 \le j \le N_y, \tag{35}$$

where $\phi_{k,j} = \phi_k(y_j)$, and $f_{i,j} = f(x_i, y_i)$. Replacing derivative ϕ' by it expression in the boundary conditions (31) and (32) one gets respectively

$$\alpha_- \sum_{k \in \mathbf{Z}} \phi_{k,N_y} \varphi_{i-k} + \beta_- \sum_{k \in \mathbf{Z}} \sum_{l=0}^{N_y} d^{(1)}_{N_y,l} \phi_{i,l} \varphi_{i-k} = g^-(x_i, -1), \qquad 0 \le i \le N_x - 1 \quad (36)$$

and

$$\alpha_+ \sum_{k \in \mathbf{Z}} \phi_{k,0} \varphi_{i-k} + \beta_+ \sum_{k \in \mathbf{Z}} \sum_{l=0}^{N_y} d^{(1)}_{0,l} \phi_{k,l} \varphi_{i-k} = g^+(x_i, 1), \qquad 0 \le i \le N_x - 1. \quad (37)$$

Now, because of the hypothesis of periodicity in the x-direction, the coefficients $\phi_{i,j}$ are periodic in the first index, the period length being N_x, i.e., $\phi_{i+N_{x,j}} = \phi_{i,j}$ (cf. Amaratunga and Williams [4]). Taking into account this periodicity, the set of equations (35) constitutes an $N_x \times (N_y + 1)$ linear system for the unknowns $\phi_{i,j}$. Then eliminating the boundary values $\{\phi_{i,0}\}_{0 \le i \le N_x - 1}$ and $\{\phi_{i,N_y}\}_{0 \le i \le N_x - 1}$ in (35) using the boundary conditions (36) and (37), we obtain an $N_x \times (N_y - 1)$ system which can be put into the matrix form

$$\mathcal{D}^c_{2,x} \Phi + \Phi \mathcal{D}^t_{2,y} - \sigma \Phi = f, \quad (38)$$

where the matrix Φ is defined as

$$\Phi_{k,j} = \Phi_{k,j}, \qquad 0 \le k \le N_x - 1, 1 \le j \le N_y - 1. \quad (39)$$

The operator $\mathcal{D}^c_{2,x}$ is the approximation of the second order derivative in the wavelet collocation method. It is a circulant operator with kernel $K^{(2)}_c = 4^J(0, \varphi''_1, \ldots, \varphi''_{N-2}, 0 \ldots, 0)$. The operator $\mathcal{D}_{2,y}$ is the discrete approximation of the second order derivative in the collocation Chebychev method when boundary conditions are included. The right hand side F in (38) contains the forcing term in the physical space and all the terms arising from the elimination of the boundary values.

3.1.2. The wavelet Galerkin/collocation Chebychev method. In the Galerkin method, Eq. (26) is projected onto the wavelet space using the basis functions as test functions. For this we multiply both sides of (26) by $\varphi(z - i)$ and integrate over $\mathbf{R}$

$$4^J \sum_{k \in \mathbf{Z}} \phi_k(y) \int_{\mathbf{R}} \varphi''(z-k)\varphi(z-i)\, dz + \sum_{k \in \mathbf{Z}} \{\phi''_k(y) - \sigma\} \int_{\mathbf{R}} \varphi(z-k)\varphi(z-i)\, dz$$
$$= \sum_{k \in \mathbf{Z}} \hat{f}_k \int_{\mathbf{R}} \varphi(z-k)\varphi(z-i)\, dz.$$

Using the orthogonality conditions $\int_{\mathbf{R}} \varphi(z-k)\varphi(z-i)\, dz = \delta_{i,k}$, one gets

$$\sum_{k \in \mathbf{Z}} 4^J \phi_k(y) \Omega_{i-k} + \phi''_i(y) - \sigma \phi_i(y) = \hat{f}_i, \qquad i \in \mathbf{Z}, \quad (40)$$

where the connection coefficients Ω_{i-k} are defined by

$$\Omega_{i-k} = \int_{\mathbf{R}} \varphi''(z-k)\varphi(z-i)\, dz. \quad (41)$$

The method for computing these coefficients was presented in Latto *et al.* [15]. The set of equations (40) form a system of the second order ordinary differential equations for the coefficients ϕ_k. Boundary conditions for Eqs. (40) are deduced from the boundary conditions (27) and (28). For this we multiply again both sides of (27) and (28) respectively by $\varphi(z-i)$ and take the integral over **R** to get respectively

$$\alpha_-\phi_i(-1) + \beta_-\phi_i'(-1) = \hat{g}_i^-, \qquad i \in \mathbf{Z} \tag{42}$$

and

$$\alpha_+\phi_i(1) + \beta_+\phi_i'(1) = \hat{g}_i^+, \qquad i \in \mathbf{Z}. \tag{43}$$

The system (40), together with the boundary conditions (42) and (43), allows the determination of the coefficients ϕ_k.

Here too, the discretisation in the non-periodic direction makes use of the Chebychev collocation method. The interval $[-1, 1]$ is discretised into the collocation points y_j, $j = 0, \ldots, N_y$, where the y_j's are defined by (33). Then writing (40) at these points one gets

$$4^J \sum_{k\in\mathbf{Z}} \phi_{k,l}\Omega_{i-k} + \sum_{j=0}^{N_y} d_{i,l}^{(2)}\phi_{k,l} - \sigma\phi_{i,j} = \hat{f}_{i,j}, \qquad 0 \le i \le N_x - 1, 0 \le j \le N_y, \tag{44}$$

where $\phi_{i,j} = \phi_i(y_j)$, $\hat{f}_{i,j} = \hat{f}_i(y_j)$.

In a similar way we replace ϕ' by its expression in the boundary conditions (42) and (43) to get respectively

$$\alpha_-\phi_{i,N_y} + \beta_- \sum_{l=0}^{N_y} d_{N_y,l}^{(1)}\phi_{i,l} = \hat{g}_{N_y}^-, \qquad 0 \le i \le N_x - 1 \tag{45}$$

and

$$\alpha_+\phi_{i,0} + \beta_+ \sum_{l=0}^{N_y} d_{0,l}^{(1)}\Phi_{i,l} = \hat{g}_0^+, \qquad 0 \le i \le N_x - 1. \tag{46}$$

Again we use the periodicity condition for the coefficients $\phi_{i,j}$ in the first direction to reduce the system (44) into a system of order $N_x \times (N_y + 1)$. Then eliminating the boundary values $\{\phi_{i,0}\}_{0\le i\le N_x-1}$ and $\{\phi_{i,N_y}\}_{0\le i\le N_x-1}$ in (44) using (45) and (46), we reduce the system to a $N_x \times (N_y - 1)$ system which in matrix form reads

$$\mathcal{D}_{2,x}^g\Phi + \Phi\mathcal{D}_{2,y}^t - \sigma\Phi = \hat{F}. \tag{47}$$

The matrix $\mathcal{D}_{2,x}^g$ is the matrix of the second derivative in the wavelet Galerkin method. It's also a circulant matrix whose kernel (i.e., the first column of the matrix) is given by

$$K_g^{(2)} = (0, \Omega_1, \ldots, \Omega_{N-2}, 0, \ldots, 0, \Omega_{2-N}, \ldots, \Omega_{-1})^t. \tag{48}$$

The matrix $\mathcal{D}_{2,y}$ is defined as before. The right hand side $\hat{F}$ contains the forcing term in the wavelet space and the quantities arising from the elimination of the boundary values.

To invert the systems (38) and (48) we first diagonalise the matrix $\mathcal{D}_{2,y}$. The diagonalisation technique is commonly associated with the spectral Chebychev approximation for solving the Helmholtz problem (Haidvogel and Zang [14]; Haldenwang *et al.* [13]; Ehrenstein and Peyret [12]).

It is known (Gottlieb and Lustman [16]) that the eigenvalues of the operator $\mathcal{D}_{2,y}$ are real negative and distinct. Thus there exists an operator $\mathcal{S}$ such that

$$\mathcal{D}_{2,y}^{t} = \mathcal{S}\Lambda\mathcal{S}^{-1}. \tag{49}$$

Λ is a diagonal matrix whose entries are the eigenvalues λ_j, $1 \leq j \leq N_y - 1$, of $\mathcal{D}_{2,y}$.

Let us consider the case of the WC/CC method. Multiplying on the right on the both sides of (38) by $\mathcal{S}$ we get

$$\mathcal{D}_{2,x}^{c}\tilde{\Phi} + \tilde{\Phi}\Lambda - \sigma\tilde{\Phi} = \tilde{F}, \tag{50}$$

where

$$\tilde{\Phi} = \Phi\mathcal{S}, \qquad \tilde{F} = F\mathcal{S}. \tag{51}$$

Now if we set

$$\tilde{\Phi}_{(j)} = \left(\tilde{\Phi}_{0,j}, \ldots, \tilde{\Phi}_{i,j}, \ldots, \tilde{\Phi}_{N_{x,j}}\right)^{t} \qquad \text{and} \qquad \tilde{F}_{(j)} = \left(\tilde{F}_{0,j}, \ldots, \tilde{F}_{i,j}, \ldots, \tilde{F}_{N_{x,j}}\right)^{t} \tag{52}$$

then the system (50) splits into $N_y - 1$ one dimensional Helmholtz problems

$$\left[\mathcal{D}_{2,x}^{c} - (\sigma - \lambda_j)\mathcal{I}\right]\tilde{\Phi}_{(j)} = \tilde{F}_{(j)}, \qquad 1 \leq j \leq N_y - 1. \tag{53}$$

In the case of the WG/CC method, applying the same process will result in a system similar to (53) with matrix $\mathcal{D}_{2,x}^{g}$ in place of $\mathcal{D}_{2,x}^{c}$ and $\tilde{\tilde{F}}_{(j)}$ instead of $\tilde{F}_{(j)}$ on the right hand side.

The systems (53) are easily solved by resorting to FFTs. Briefly, let $H_j = \mathcal{D}_{2,x}^{c} - (\sigma - \lambda_j)$ be the one dimensional Helmholtz operator. Then H_j is a circulant operator and the system (53) may be put in a convolution form

$$K_{H_j} * \tilde{\Phi}_{(j)} = \tilde{F}_{(j)}, \tag{54}$$

where the kernel K_{H_j} is the first column of the operator H_j. Then taking the Fourier transform of (54) one gets

$$\mathcal{F}_k\left(K_{H_j}\right) \cdot \mathcal{F}_k\left(\tilde{\Phi}_{(j)}\right) = \mathcal{F}_k\left(\tilde{F}_{(j)}\right). \tag{55}$$

The wavelet coefficients in the Fourier space $\mathcal{F}_k(\tilde{\Phi}_{(j)})$ can be eliminated from (55) allowing one to work only with the Fourier coefficients of quantities in the physical space. For this we notice first the relations (22) and (23) are also valid for $\tilde{\Phi}_{(j)}$ and its Fourier spectrum $\mathcal{F}_k(\tilde{\Phi}_{(j)})$. Thus substituting in (55) $\mathcal{F}_k(\tilde{\Phi}_{(j)})$ with its expression $\mathcal{F}_k(\tilde{U}_{(j)})/\mathcal{F}_k(K_{\varphi})$ we get

$$\mathcal{F}_k\left(K_{H_j}^{c}\right) \cdot \mathcal{F}_k\left(\tilde{U}_{(j)}\right) = \mathcal{F}_k\left(F_{(j)}\right), \tag{56}$$

where $\mathcal{F}_k(K^c_{H_j}) = \mathcal{F}_k(K_{H_j})/\mathcal{F}_k(K_\varphi)$. The quantities $\mathcal{F}_k(\tilde{U}_{(j)})$ are the Fourier coefficients of the vector $\tilde{U}_{(j)}$, which is obtained from the matrix U of the physical values after applying the process (51) and (52). In the WG/CC method both coefficients $\mathcal{F}_k(\tilde{\Phi}_{(j)})$ and $\mathcal{F}_k(\tilde{\tilde{F}}_{(j)})$ have to be eliminated. It results then in a system analogous to (56) with $\mathcal{F}_k(K^g_{H_j}) = \mathcal{F}_k(K_{H_j})$ in place of $\mathcal{F}_k(K^c_{H_j})$.

Remark. For Neumann boundary conditions, one eigenvalue in the spectrum of $\mathcal{D}_{2,y}$ is equal to 0. Thus in the case of the Poisson equation ($\sigma = 0$) with Neumann boundary conditions, the first coefficient $\mathcal{F}_0(K_{H_j})$ in the Fourier spectrum of K_{H_j} is zero in both the WC/CC and WG/CC method. In order to avoid a division by zero in (56), we set $\mathcal{F}_0(\tilde{U}_{(j)}) = 0$.

4. SOLUTION METHOD FOR THE STOKES-LIKE PROBLEM

In this section, we present the numerical method for solving the stationary Stokes problem (12)–(13) with boundary conditions (10)–(11) and periodicity in the first direction. For simplicity of notation we drop the superscript $n+1$ on the variables and rewrite the problem (12)–(13) as

$$\nabla^2\omega - \sigma\omega = f \qquad \text{in } D \tag{57}$$

$$\nabla^2\psi + \omega = 0 \qquad \text{in } D \tag{58}$$

together with boundary conditions

$$\psi = g, \qquad \text{on } \Gamma \tag{59}$$

$$\partial_y\psi = h, \qquad \text{on } \Gamma \tag{60}$$

and periodicity conditions in the direction of x. The domain D and the confined boundary Γ are defined by (6) and (7), respectively. As was pointed out earlier, the main difficulty in solving problem (57)–(60) arises from the lack of boundary conditions for the vorticity while 2 boundary conditions are prescribed for the stream function.

To solve the above problem we look for Dirichlet boundary conditions μ on ω such that if a pair of functions (ω, ψ) which are periodic in x is a solution of

$$\begin{cases} \nabla^2\omega - \sigma\omega = f, & \text{in } D \\ \omega = \mu, & \text{on } \Gamma \end{cases} \tag{61}$$

and

$$\begin{cases} \nabla^2\psi + \omega = 0, & \text{in } D \\ \psi = g, & \text{on } \Gamma \end{cases} \tag{62}$$

then the function ψ satisfies the Neumann boundary condition (60). This will be done by resorting to the influence matrix method.

We first discretise the domain D and the boundary Γ into collocation points

$$D_c = \{(x_i, y_j), 0 \le i \le N_x - 1, 1 \le j \le N_y\}$$

and

$$\Gamma_c = \{(x_i, -1), (x_i, 1), 0 \le i \le N_x - 1\}$$

and let N_Γ be the cardinality of Γ_c: $N_\Gamma = 2^{J+1}$. The essence of the influence matrix method lies in the superposition principle for linear problems. We are looking for the pair functions (ω, ψ) in the form

$$(\omega, \psi) = (\tilde{\omega}, \tilde{\psi}) + \sum_{j=1}^{N_\Gamma} \mu_j (\bar{\omega}_j, \bar{\psi}_j), \tag{63}$$

where $(\tilde{\omega}, \tilde{\psi})$ is a solution of

$$\begin{cases} \nabla^2 \tilde{\omega} - \sigma \tilde{\omega} = \tilde{f}, & \text{in } D_c \\ \tilde{\omega} = 0, & \text{on } \Gamma_c \end{cases} \tag{64}$$

$$\begin{cases} \nabla^2 \tilde{\psi} + \tilde{\omega} = 0, & \text{in } D_c \\ \tilde{\psi} = g, & \text{on } \Gamma_c \end{cases} \tag{65}$$

and for $1 \le j \le N_\Gamma$ each pair of functions $(\bar{\omega}_j, \bar{\psi}_j)$ is associated with a point p_j on the boundary Γ_c and is defined by the homogeneous problem

$$\begin{cases} \nabla^2 \bar{\omega}_j - \sigma \bar{\omega}_j = 0 & \text{in } D_c \\ \bar{\omega}_j(p_k) = \delta_{jk} & \text{on } \Gamma_c \end{cases} \tag{66}$$

$$\begin{cases} \nabla^2 \bar{\psi}_j + \bar{\omega}_j = 0 & \text{in } D_c \\ \bar{\psi}_j = 0 & \text{on } \Gamma_c \end{cases} \tag{67}$$

such that by construction, the functions ω and ψ considered in the decomposed form (63) are solutions of Eqs. (57) and (58) and in addition the function ψ satisfies the Dirichlet boundary conditions (59). The coefficients μ_k are then determined by demanding that the function ψ satisfies the Neumann boundary conditions (60). By writing this condition on ψ considered in the form (63) we get

$$\sum_{j=1}^{N_\Gamma} \mu_j \partial_y \bar{\psi}_j(p_k) = h(p_k) - \partial_y \tilde{\psi}(p_k). \tag{68}$$

This constitutes a system of N_Γ equations for the N_Γ coefficients μ_k and the system may be put into the matrix form

$$\mathcal{M}\Theta = \mathcal{R}, \tag{69}$$

where $\Theta = (\mu_0, \ldots, \mu_i, \ldots, \mu_{N_\Gamma})^t$. The capacitance matrix $\mathcal{M}$ and the right hand side of (69) are defined by

$$\mathcal{M}_{k,j} = \partial_y \tilde{\psi}_j(p_k), \qquad \mathcal{R}_k = h(p_k) - \partial_y \tilde{\psi}(p_k). \tag{70}$$

Thus the solution of the problem amounts to solving a series of Helmholtz and Poisson problems with periodic and Dirichlet boundary conditions. The numerical resolution of these problems is done either by the WC/CC or the WG/CC method presented in Section 2.

5. NUMERICAL RESULTS

5.1. *Numerical Results for the Helmholtz Problem*

We present here the numerical results obtained by applying the previous methods to solving the Helmholtz equation. The methods are validated against the following analytical solution

$$u_{ex}(x, y) = (1 - y^2)^2\, e^{1+y}\, e^{150(x-1/2)^2}. \tag{71}$$

The forcing term in Eqs. (14)–(16) is obtained from the solution (71). Various numerical tests have been conducted by changing the degree of the Daubechies wavelet N and the number J of the scale. The number of collocation in the y-direction is kept constant and large enough ($N_y = 24$) so that the error in the wavelet approximation is greater than the error in the Chebychev approximation. The results obtained by using various kinds of boundary conditions are similar. Thus we present only the results obtained with Dirichlet boundary conditions: i.e., by taking $\alpha_- = \alpha_+ = 1$ and $\beta_- = \beta_+ = 0$ in (15)–(16).

Tables I and II show the relative pointwise error on the numerical solution by the WC/CC and WG/CC methods, respectively. It can be seen that for small degree of the Daubechie scaling function, the results obtained by the WC/CC method are poor (Table I). The convergence of the error is too slow. On the other hand, good decrease of the error is observed even for small degree of the scaling function when the WG/CC method is used (Table II). This is normal since in the WC/CC one needs the basis functions to be sufficiently regular while in the WG/CC one asks only the basis functions to belong to $\mathbf{L}^2(\mathbf{R})$. The regularity of the Daubechies scaling function increases with its degree. For $N = 14$ the basis functions are at least in $C^2(\mathbf{R})$ (cf. Daubechies [1]). It can be seen in Tables I and II that when the degree is taken large enough to ensure sufficient regularity for the scaling function, the decrease of the error in both methods is comparable to that usually observed in spectral methods.

TABLE I
Pointwise Error in the WC/CC Solution of the Helmholtz Problem with Dirichlet Boundary Conditions, (a) σ = 0, (b) σ = 1000

	J	5	6	7	8	9
8	(a)	5.224×10^{-1}	2.378×10^{-1}	9.839×10^{-2}	2.975×10^{-2}	7.957×10^{-3}
	(b)	4.914×10^{-1}	1.840×10^{-1}	7.626×10^{-2}	2.300×10^{-2}	6.154×10^{-3}
10	(a)	8.563×10^{-2}	1.622×10^{-2}	2.298×10^{-3}	2.918×10^{-4}	3.770×10^{-5}
	(b)	4.410×10^{-2}	8.235×10^{-3}	1.151×10^{-3}	1.467×10^{-4}	1.863×10^{-5}
12	(a)	5.279×10^{-2}	5.375×10^{-3}	3.528×10^{-4}	2.217×10^{-5}	1.289×10^{-6}
	(b)	2.735×10^{-2}	2.892×10^{-3}	1.926×10^{-4}	1.267×10^{-5}	7.413×10^{-7}
14	(a)	2.507×10^{-2}	1.181×10^{-3}	3.950×10^{-5}	1.261×10^{-6}	3.965×10^{-8}
	(b)	1.344×10^{-2}	7.083×10^{-4}	2.390×10^{-5}	7.634×10^{-7}	2.400×10^{-8}
16	(a)	7.730×10^{-3}	2.383×10^{-4}	4.460×10^{-6}	7.406×10^{-8}	2.209×10^{-9}
	(b)	4.314×10^{-3}	1.478×10^{-4}	2.834×10^{-6}	4.680×10^{-8}	9.737×10^{-10}
18	(a)	1.156×10^{-3}	1.580×10^{-5}	1.785×10^{-10}	1.583×10^{-9}	4.372×10^{-11}
	(b)	7.577×10^{-4}	1.017×10^{-5}	1.199×10^{-7}	1.065×10^{-10}	9.380×10^{-12}

TABLE II
Pointwise Error in the WG/CC Solution to the Helmholtz Problem with Dirichlet Boundary Conditions, (a) $\sigma = 0$, (b) $\sigma = 1000$

	J	5	6	7	8	9
6	(a)	2.164×10^{-2}	2.061×10^{-3}	1.422×10^{-4}	9.084×10^{-6}	5.698×10^{-7}
	(b)	1.133×10^{-2}	1.111×10^{-3}	7.754×10^{-5}	4.969×10^{-6}	3.124×10^{-7}
8	(a)	4.017×10^{-3}	1.159×10^{-4}	2.114×10^{-6}	3.297×10^{-8}	6.108×10^{-9}
	(b)	2.384×10^{-3}	7.231×10^{-5}	1.338×10^{-6}	2.181×10^{-8}	3.093×10^{-10}
10	(a)	1.139×10^{-3}	1.093×10^{-5}	5.397×10^{-8}	9.430×10^{-10}	3.398×10^{-10}
	(b)	7.347×10^{-4}	7.467×10^{-6}	3.760×10^{-8}	1.467×10^{-10}	8.124×10^{-11}
12	(a)	4.088×10^{-4}	1.386×10^{-6}	2.624×10^{-9}	7.384×10^{-10}	8.671×10^{-10}
	(b)	2.795×10^{-4}	1.006×10^{-6}	1.564×10^{-9}	1.580×10^{-10}	1.522×10^{-10}
14	(a)	1.724×10^{-4}	2.161×10^{-7}	3.381×10^{-10}	1.226×10^{-9}	4.978×10^{-9}
	(b)	1.229×10^{-4}	1.638×10^{-7}	1.089×10^{-10}	1.226×10^{-9}	4.605×10^{-11}

5.2. *Numerical Results for the Stationary Stokes-like Problem*

We present the numerical results obtained using the influence matrix method for the solution of the stationary Stokes problem. The following analytical solution is considered to validate the method

$$\psi_{ex}(x, y) = (1 - y^2)^2\, e^{1+y}\, e^{150(x-1/2)}, \qquad \Omega_{ex} = -\nabla^2 \psi_{ex} \tag{72}$$

which satisfies the no-slip conditions on Γ (i.e., $h = g = 0$ in (59) and (60)). The forcing term in (57) is deduced from the analytical solution (72).

Tables III and IV show the condition number of the capacitance matrix in the WC/CC and WG/CC methods, respectively. It can be seen that the condition number deteriorates with decreasing scales but is improved by increasing the degree of regularity of the scaling function. However, the conditioning of the capacitance matrix doesn't seem to influence too much the numerical results. In the WC/CC method the degree of the scaling function is taken large enough to ensure a minimum of regularity. Tables V and VI, VII and VIII show the pointwise errors on the vorticity function in the WC/CC and WG/CC methods, respectively. The numerical results are obtained for 2 values of the parameter σ: value 10 corresponding to a large time step and value 1000 corresponding to a small time step in the

TABLE III
Condition Number of the Capacitance Matrix in the WC/CC Method, (a) $\sigma = 1$, (b) $\sigma = 1000$

	J	4	5	6	7	8
12	(a)	1.091×10^{-2}	3.548×10^{-3}	4.755×10^{-4}	3.985×10^{-5}	2.532×10^{-6}
	(b)	2.128×10^{-1}	7.083×10^{-2}	1.059×10^{-2}	9.944×10^{-4}	6.097×10^{-5}
14	(a)	1.332×10^{-2}	4.830×10^{-3}	9.147×10^{-4}	9.081×10^{-5}	5.978×10^{-6}
	(b)	2.666×10^{-1}	1.119×10^{-1}	2.202×10^{-2}	2.154×10^{-3}	1.534×10^{-4}
16	(a)	1.529×10^{-2}	6.333×10^{-3}	1.183×10^{-3}	1.245×10^{-4}	8.828×10^{-6}
	(b)	2.960×10^{-1}	1.292×10^{-1}	2.856×10^{-2}	2.960×10^{-3}	2.152×10^{-4}
18	(a)	1.965×10^{-2}	8.476×10^{-3}	3.208×10^{-2}	1.480×10^{-4}	1.082×10^{-5}
	(b)	2.980×10^{-1}	1.383×10^{-1}	1.398×10^{-3}	3.484×10^{-3}	2.575×10^{-4}

TABLE IV
Condition Number of the Capacitance Matrix in the WG/CC Method, (a) $\sigma = 1$, (b) $\sigma = 1000$

	J	4	5	6	7	8
6	(a)	9.948×10^{-3}	3.888×10^{-3}	6.074×10^{-4}	5.322×10^{-5}	1.321×10^{-6}
	(b)	2.062×10^{-1}	8.592×10^{-2}	1.429×10^{-2}	1.272×10^{-3}	3.059×10^{-5}
8	(a)	1.470×10^{-2}	4.897×10^{-3}	8.786×10^{-4}	8.096×10^{-5}	6.356×10^{-6}
	(b)	2.657×10^{-1}	1.064×10^{-1}	2.024×10^{-2}	1.940×10^{-3}	1.372×10^{-4}
12	(a)	1.544×10^{-2}	5.355×10^{-3}	1.022×10^{-3}	9.747×10^{-5}	7.784×10^{-6}
	(b)	2.787×10^{-1}	1.162×10^{-1}	2.248×10^{-2}	2.335×10^{-3}	1.672×10^{-4}

TABLE V
Pointwise Error on ω in the Solution of the Stationary Stokes Problem, with the WC/CC Method, (a) $\sigma = 0$, (b) $\sigma = 1000$

	J	5	6	7	8
12	(a)	2.129×10^{-1}	2.534×10^{-2}	1.734×10^{-3}	1.097×10^{-4}
	(b)	1.281×10^{-1}	1.592×10^{-2}	1.099×10^{-3}	6.979×10^{-5}
14	(a)	1.074×10^{-1}	7.076×10^{-3}	2.388×10^{-4}	7.631×10^{-6}
	(b)	6.787×10^{-2}	4.728×10^{-3}	7.891×10^{-5}	5.149×10^{-6}
16	(a)	3.964×10^{-2}	1.535×10^{-3}	3.031×10^{-5}	5.047×10^{-7}
	(b)	2.411×10^{-2}	1.047×10^{-3}	2.104×10^{-5}	1.318×10^{-7}
18	(a)	8.573×10^{-3}	1.099×10^{-4}	1.391×10^{-6}	1.252×10^{-8}
	(b)	6.065×10^{-3}	7.655×10^{-5}	9.979×10^{-7}	9.045×10^{-9}

TABLE VI
Pointwise Error on ψ in the Solution of the Stationary Stokes Problem, with the WC/CC Method, (a) $\sigma = 0$, (b) $\sigma = 1000$

	J	5	6	7	8
12	(a)	9.995×10^{-2}	1.701×10^{-2}	7.050×10^{-4}	4.426×10^{-5}
	(b)	7.630×10^{-2}	8.240×10^{-3}	5.451×10^{-4}	3.429×10^{-5}
14	(a)	5.185×10^{-2}	2.374×10^{-3}	7.891×10^{-5}	2.519×10^{-6}
	(b)	3.891×10^{-2}	1.898×10^{-3}	6.307×10^{-5}	2.022×10^{-6}
16	(a)	1.629×10^{-2}	4.776×10^{-4}	8.935×10^{-6}	1.484×10^{-7}
	(b)	1.257×10^{-2}	3.870×10^{-4}	7.295×10^{-6}	2.470×10^{-9}
18	(a)	2.312×10^{-3}	3.156×10^{-5}	3.568×10^{-7}	3.161×10^{-9}
	(b)	1.917×10^{-3}	2.595×10^{-5}	2.981×10^{-7}	2.645×10^{-9}

TABLE VII
Pointwise Error on ω in the Solution of the Stationary Stokes Problem, with the WG/CC, (a) $\sigma = 0$, (b) $\sigma = 1000$

	J	5	6	7	8
6	(a)	8.433×10^{-2}	9.602×10^{-3}	6.936×10^{-4}	4.473×10^{-5}
	(b)	5.273×10^{-2}	6.058×10^{-3}	4.400×10^{-4}	2.844×10^{-5}
8	(a)	2.155×10^{-2}	7.525×10^{-4}	1.438×10^{-5}	2.361×10^{-7}
	(b)	1.426×10^{-2}	2.310×10^{-4}	9.990×10^{-6}	1.646×10^{-7}
10	(a)	7.580×10^{-3}	9.013×10^{-5}	4.727×10^{-7}	1.663×10^{-9}
	(b)	5.272×10^{-3}	6.569×10^{-5}	3.494×10^{-7}	3.586×10^{-10}
12	(a)	3.204×10^{-3}	1.378×10^{-5}	2.114×10^{-8}	8.555×10^{-10}
	(b)	2.315×10^{-3}	1.048×10^{-5}	3.065×10^{-8}	3.586×10^{-10}
14	(a)	1.539×10^{-3}	2.507×10^{-6}	1.308×10^{-9}	2.079×10^{-10}
	(b)	1.145×10^{-3}	1.970×10^{-6}	9.481×10^{-10}	2.079×10^{-10}

temporal discretisation. The results don't depend too much on the value of σ. As in the case of the Helmholtz problem, a spectral accuracy is reached provided the scaling function is sufficiently regular.

5.3. *Numerical Results for the Instationary Stokes Problem*

We present here the numerical results obtained for the unsteady Stokes problem. After discretising the problem (1)–(4) using the scheme (8)–(11), the resulting stationary Stokes-like problems is solved at each time step by the methods previously presented. The algorithm is first validated against the analytical solution

$$\psi_{ex}(x, y, t) = \cos(t)\cos(2\pi x)(1 - y^2)^2\, e^{1+y}, \qquad \omega_{ex} = -\nabla^2 \psi_{ex}. \tag{73}$$

Figure 1 shows the results obtained for the stream function. The results for the vorticity are similar. The number of collocation points in the y-direction is still $N_y = 24$. These results are obtained by taking the degree of the scaling function to be $N = 8$ and the scale of resolution in x is $J = 5$ (i.e., $N_x = 32$) in the case of the WG/CC method and $N = 12$ and

TABLE VIII
Pointwise Error on ψ in the Solution of the Stationary Stokes Problem, with the WG/CC Method, (a) $\sigma = 0$, (b) $\sigma = 1000$

	J	5	6	7	8
6	(a)	4.060×10^{-2}	3.141×10^{-3}	2.836×10^{-4}	1.813×10^{-5}
	(b)	3.124×10^{-2}	4.074×10^{-3}	2.192×10^{-4}	1.403×10^{-5}
8	(a)	7.795×10^{-3}	5.166×10^{-4}	4.223×10^{-6}	6.760×10^{-8}
	(b)	6.232×10^{-3}	1.877×10^{-4}	3.449×10^{-6}	5.477×10^{-8}
10	(a)	2.234×10^{-3}	2.183×10^{-5}	1.076×10^{-7}	7.543×10^{-10}
	(b)	1.841×10^{-3}	1.837×10^{-5}	9.150×10^{-8}	8.817×10^{-10}
12	(a)	8.060×10^{-4}	2.769×10^{-6}	5.224×10^{-9}	1.419×10^{-9}
	(b)	6.793×10^{-4}	2.389×10^{-6}	4.178×10^{-9}	8.817×10^{-10}
14	(a)	3.407×10^{-4}	4.320×10^{-7}	3.989×10^{-10}	7.556×10^{-10}
	(b)	2.921×10^{-4}	3.797×10^{-7}	3.449×10^{-10}	1.156×10^{-10}

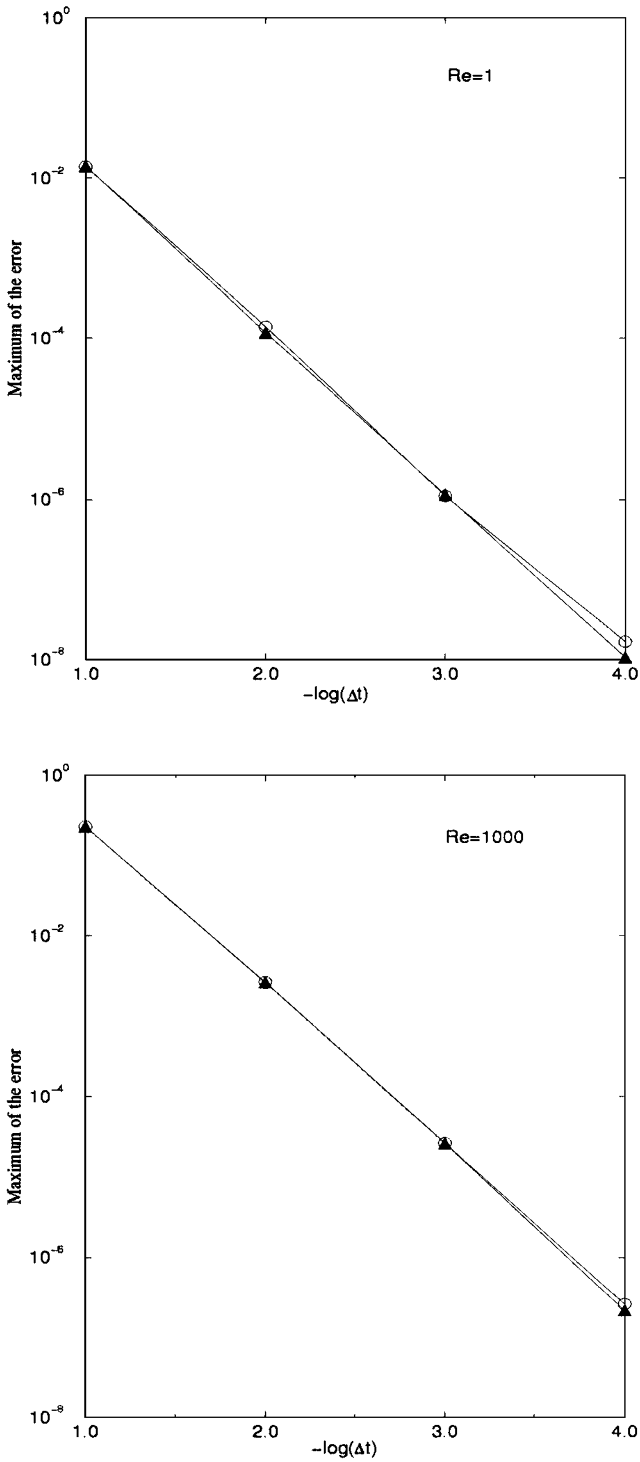

FIG. 1. Maximum of the pointwise error in time of ψ with respect to the time step Δt, with the WC/CC method (circles) and the WG/CC method (triangles).

$J = 6$ for the WC/CC method. These choices of spatial parameters are such that the spatial error is always smaller than the error in time. Clearly the results in Fig. 1 show a $\mathcal{O}(\Delta t^2)$ decrease of the error. A good stability of the method is also observed.

6. CONCLUSION

A mixed wavelet/spectral Chebychev method has been developed for solving the 2D Stokes equations with periodicity condition in one direction. In the periodic direction the approximation is done on the basis of the translates and dilations of the Daubechies scaling function. The discretistion is done either in the wavelet collocation method or the wavelet galerkin method. Then in the non-periodic direction the collocation Chebychev method is used. A capacitance matrix method has been implemented to handle the boundary conditions. This leads to a series of Helmholtz systems which are efficiently inverted using the following ingredients:

- Diagonalisation in the non-periodic direction
- FFTs for inverting the operators in the wavelet space.

Numerical tests conducted on analytical solutions show that the method is stable and spectrally accurate with regard to the degree of regularity of the Daubechies scaling function. The method can be extended to the Navier–Stokes equations by using an appropriate treatment of the non-linear terms.

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